Dynamique de type Pisot et quasicristaux

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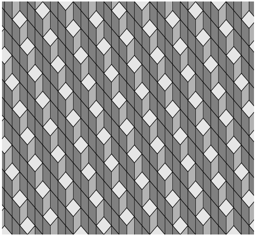
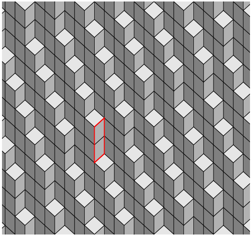
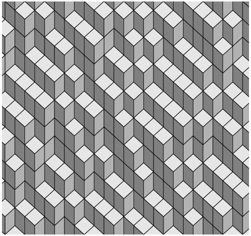
Femmes & Mathématiques-Colloque des 30 ans

Toward long-range aperiodic order







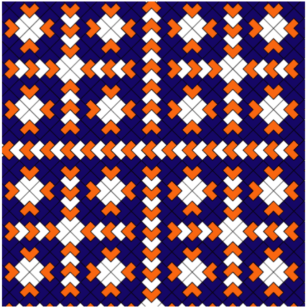
What is meant by quasiperiodicity?

The objects under consideration

- Infinite words (sequences with values in a finite alphabet)

$$abaababaabaababaababaabaababaabaababaababaa\cdots$$

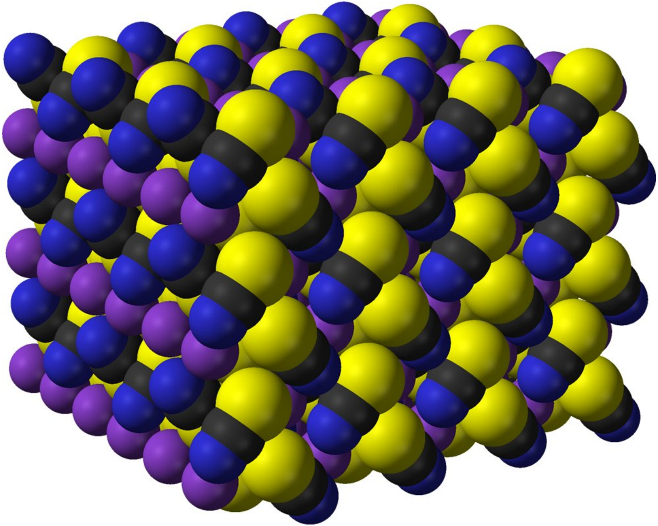
- Tilings



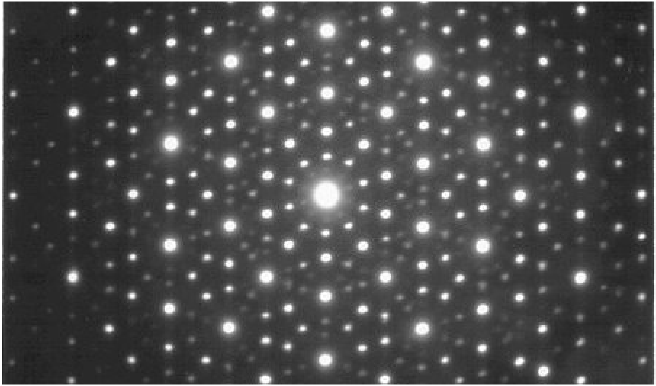
A tiling of the plane is a collection of tiles that covers the plane with no overlaps

Tilings and quasicrystals

Crystals and periodicity

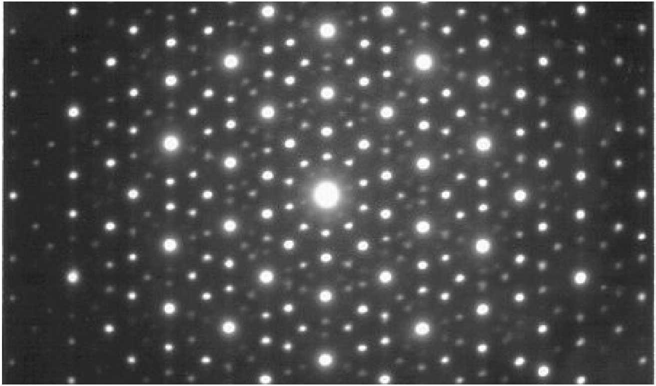


Crystals and diffraction



Periodicity $\Rightarrow$ discrete diffractogram

Crystals and diffraction



Periodicity $\not\Leftarrow$ discrete diffractogram

Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order

(cf. Aperiodic tilings [Wang'61, Berger'66, Robinson'71,...])

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- Quasicrystals produce a discrete diffraction diagram (=order)
- Diffraction comes from regular spacing and local interactions of the point set Λ (consider the relative positions $\Lambda - \Lambda$)

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"His discovery was extremely controversial. In the course of defending his findings, he was asked to leave his research group. However, his battle eventually forced scientists to reconsider their conception of the very nature of matter."

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Which mathematical models for quasicrystals?

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Which mathematical models for quasicrystals?

There are mainly two methods for producing quasicrystals

- Substitutions
- Cut and project schemes

[WHAT IS.. a Quasicrystal? M. Senechal]

Which models for quasicrystals?

Aperiodic mosaics, such as those found in the medieval Islamic mosaics of the Alhambra Palace in Spain and the Darb-i Imam Shrine in Iran, have helped scientists understand what quasicrystals look like at the atomic level. In those mosaics, as in quasicrystals, the patterns are regular - they follow mathematical rules - but they never repeat themselves.

When scientists describe Shechtman's quasicrystals, they use a concept that comes from mathematics and art: the golden ratio.

Tilings and art

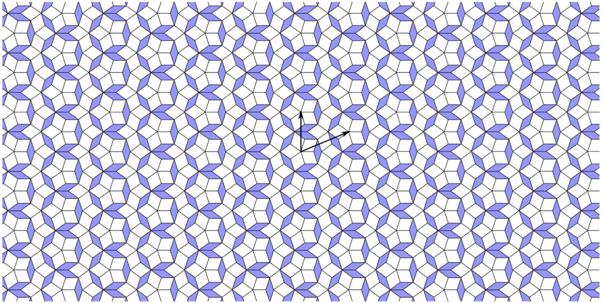
Tilings and art

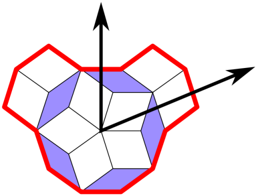
Tilings and design

Floors

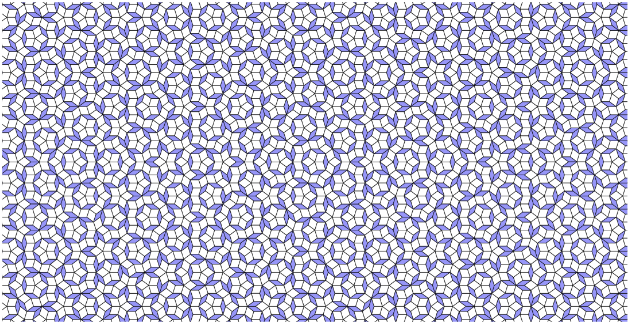
© Thomas Fernique – Image des mathématiques

A periodic tiling





A quasiperidoic tiling

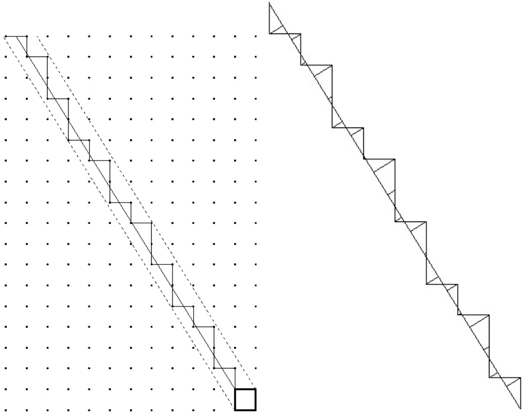


Every finite patch can be found
in any sufficiently large patch

Cut and project schemes

Projection of a “plane” slicing through a higher dimensional lattice

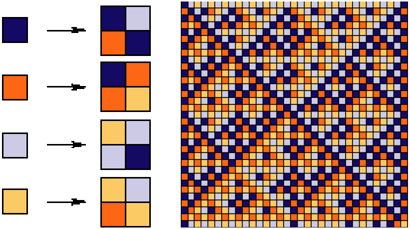
- The order comes from the lattice structure
- The nonperiodicity comes from the irrationality of the normal vector of the “plane”



Substitutions

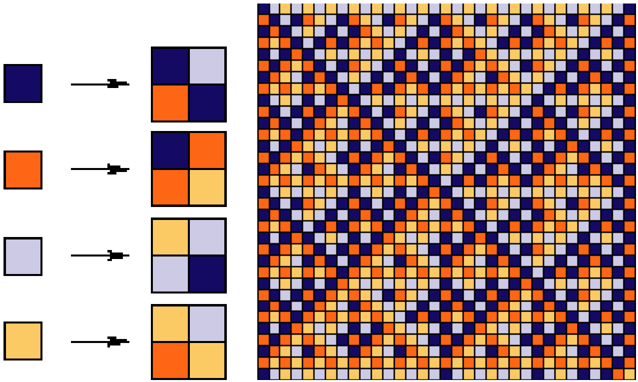
Substitutions

- Substitutions on words and symbolic dynamical systems
- Substitutions on tiles : inflation/subdivision rules, tilings and point sets



- Tilings Encyclopedia http://tilings.math.uni-bielefeld.de/

[E. Harriss, D. Frettlöh]



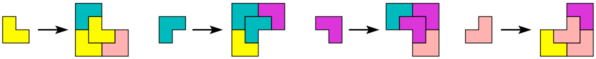
Substitutions and tilings

Principle One takes

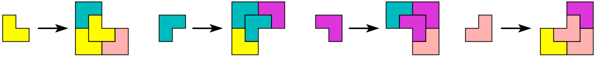
- a finite number of prototiles $\{T_1, T_2, \ldots, T_m\}$
- an expansive transformation Q
- a rule that allows one to divide each QT_i into copies of the $T_1, T_2, \ldots, T_m$

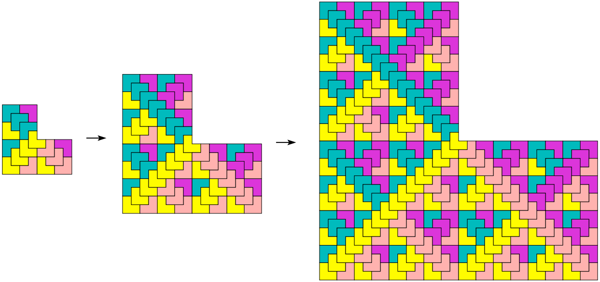
A substitution is a simple production method that allows one to construct infinite tilings using a finite number of tiles

Example

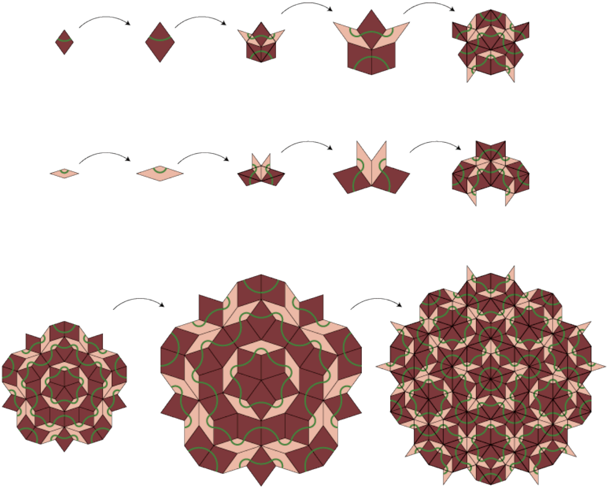


The chair tiling





Substitutions allow the construction of aperiodic tilings



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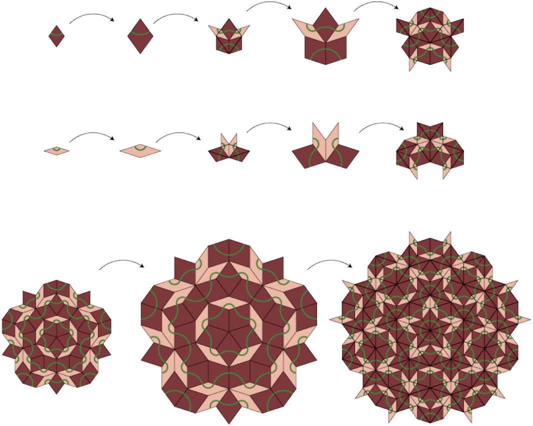

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A substitution on words : the Fibonacci substitution

Definition A substitution σ is a morphism of the free monoid

Positive morphism of the free group, no cancellations

Example

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 1$$

1
12
121
12112
12112121

$$\sigma^{\infty}(1) = 1211212112112 12\cdots$$

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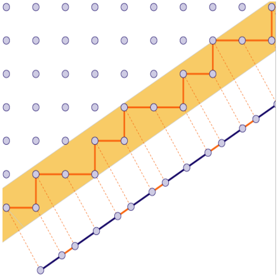

The Fibonacci word yields a quasicrystal

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Why the terminology Fibonacci word?

$$\sigma^{n+1}(1) = \sigma^{n}(12) = \sigma^{n}(1)\sigma^{n}(2)$$

$$\sigma^{n}(2) = \sigma^{n-1}(1)$$

$$\sigma^{n+1}(1) = \sigma^{n}(1)\sigma^{n-1}(1)$$

The length of the word $\sigma^{n}(1)$ satisfies the Fibonacci recurrence

Which substitutions do generate quasicrystals?

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$\omega = abaababaabaababaababaabaababaabaababaababaa \cdots$$

- There is a simple algorithmic way to construct it

  (cf. Kolmogorov complexity)

  But not all substitutions do produce quasicrystals

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$\omega = abaababaabaababaababaabaababaabababaa \cdots$$

- There are few local configurations = factors

  A factor is a word made of consecutive occurrences of letters

  ab is a factor, *bb* is not a factor of the Fibonacci word

  But

  $$\cdots aaaaaaaaaaaabaaaaaaaaaaa \cdots$$

  has as many factors of length n as

  $$\cdots abaababaabaababaababaabaababaabababaa \cdots$$

  The Fibonacci word has $n+1$ factors of length n

  But words with $2n+1$ factors of length n are not all quasicrystals!

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$\omega = abaababaabaababaababaabaababaabaababaababaa \cdots$$

- Consider densities of occurrences of factors

  Symbolic discrepancy

  $$\Delta_N = \max_{i \in \mathcal{A}} \left| |\omega_0 \omega_1 \dots \omega_{N-1}|_i - N \cdot f_i \right|$$

  if each letter i has density f_i in ω

  $$f_i = \lim_{N \to \infty} \frac{|\omega_0 \cdots \omega_{N-1}|_i}{N}$$

  The Fibonacci word has bounded symbolic discrepancy

  (cf. good equidistribution properties for real numbers having bounded partial quotients)

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$\omega = abaababaabaababaababaabaababaabaababaababaa \cdots$$

- Let $\alpha \in [0,1]$. We consider the Kronecker sequence $(\{n\alpha\})_n$ associated with the translation over $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$R_\alpha \colon \mathbb{T} \mapsto \mathbb{T},\ x \mapsto x + \alpha$$

Discrepancy

$$\begin{aligned} \Delta_N &= \sup_{I \text{ interval}} \left|\operatorname{Card}\{0 \leq n \leq N; \{n\alpha\} \in I\} - N \cdot \mu(I)\right| \\ &= \sup_{I \text{ interval}} \left|\operatorname{Card}\{0 \leq n \leq N; R_\alpha^n(0) \in I\} - N \cdot \mu(I)\right| \end{aligned}$$

Symbolic discrepancy

$$\Delta_N = \max_{i \in \mathcal{A}} \left||\omega_0\omega_1 \ldots \omega_{N-1}|_i - N \cdot f_i\right|$$

The Tribonacci substitution [Rauzy'82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$\sigma^\infty(1) : \quad 1213121121312121 3 \cdots$$

Its incidence matrix is $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The number of i in $\sigma^n(j)$ is given by $M_\sigma^n[i, j]$

Its characteristic polynomial is $X^3 - X^2 - X - 1$

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It is primitive: there exists a power of M_σ which contains only positive entries

$\rightsquigarrow$ Perron-Frobenius theory

one expanding eigendirection

a contracting eigenplane

Pisot number

Pisot-Vijayaraghavan number An algebraic integer is a Pisot number if its algebraic conjugates λ (except itself) satisfy

$$|\lambda| < 1$$

Pisot substitution σ is primitive and its Perron–Frobenius eigenvalue (for its incidence matrix) is a Pisot number

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Pisot + Perron-Frobenius $\rightsquigarrow$ one expanding eigendirection
a contracting eigenplane

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Fact Words generated by Pisot substitutions have bounded symbolic discrepancy

$$\Delta_N = \max_{i \in \mathcal{A}} \left| |\omega_0 \omega_1 \ldots \omega_{N-1}|_i - N \cdot f_i \right|$$

$$\text{with } f_i = \lim_{N \to \infty} \frac{|\omega_0 \cdots \omega_{N-1}|_i}{N}$$

Substitutive dynamical systems

Let σ be a primitive substitution over $\mathcal{A}$.
Let $\omega = (\omega_n)$ with $\sigma(\omega) = \omega$ be an infinite word generated by σ.
Let S be the shift

$$S((\omega_n)_n) = (\omega_{n+1})_n$$

$$S(abaababaa\cdots) = baababaa\cdots$$

The symbolic dynamical system generated by σ is (X_σ, S)

$$X_\sigma := \overline{\{S^n(\omega);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

This is the set of infinite words whose factors are also factors of ω

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

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Eigenvalue Let (X, T) be a topological dynamical system

T is a homeomorphism acting on the compact space X

Example $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ $\qquad R_\alpha \colon \mathbb{T} \mapsto \mathbb{T},\ x \mapsto x + \alpha$

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

A non-zero continuous function $f \in \mathcal{C}(X)$ with complex values is an eigenfunction for T if there exists $\lambda \in \mathbb{C}$ such that

$$\forall x \in X, \ f(Tx) = \lambda f(x)$$

Discrete spectrum (X, T) is said to have pure discrete spectrum if its eigenfunctions span $\mathcal{C}(X)$

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

Example

$$R_\alpha \colon \mathbb{T}/\mathbb{Z} \to \mathbb{T}/\mathbb{Z},\ x \mapsto x + \alpha$$

$$f_k \colon x \mapsto e^{2i\pi kx},\ f_k \circ R_\alpha = e^{2i\pi k\alpha} f_k$$

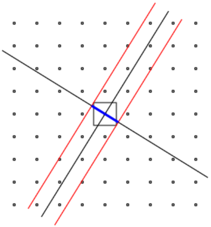
Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

Theorem [Von Neumann] Any invertible and minimal topological dynamical system minimal with topological discrete spectrum is isomorphic to a minimal translation on a compact abelian group

Example In the Fibonacci case $\sigma\colon 1 \mapsto 12, 2 \mapsto 1$
(X_σ, S) is measure-theoretically isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$

$$\begin{tikzcd} X_\sigma \arrow[r, "S"] \arrow[d] & X_\sigma \arrow[d] \\ \mathbb{T} \arrow[r, "R_\alpha"'] & \mathbb{T} \end{tikzcd}$$

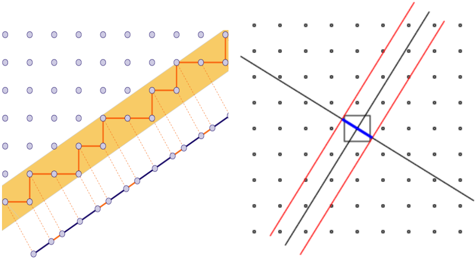


The Pisot substitution conjecture

Substitutive structure $+$ Algebraic assumption (Pisot)

$=$ Order (discrete spectrum)

Discrete spectrum $=$ translation on a compact group



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Question Under which conditions is it possible to give a geometric representation of a substitutive dynamical system as a translation on a compact abelian group? (discrete spectrum)

Substitutive dynamical systems

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The symbolic dynamical system generated by σ is (X_σ, S)

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The Pisot substitution conjecture Dates back to the 80's

[Bombieri-Taylor, Rauzy,Thurston]

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

Substitutive dynamical systems

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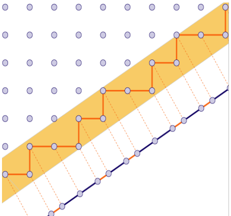
Example In the Fibonacci case

$$\sigma\colon 1 \mapsto 12, 2 \mapsto 1$$

(X_σ, S) is measure-theoretically isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$

$$R_{\frac{1+\sqrt{5}}{2}} : x \mapsto x + \frac{1+\sqrt{5}}{2} \bmod 1$$



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The Pisot substitution conjecture

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

The conjecture is proved for two-letter alphabets

[Host, Barge-Diamond, Hollander-Solomyak]

Tribonacci's substitution [Rauzy '82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1 \qquad \begin{array}{ccc} X_\sigma & \xrightarrow{S} & X_\sigma \\ \downarrow & & \downarrow \\ \mathbb{T}^2 & \xrightarrow[+(1/\beta, 1/\beta^2)]{} & \mathbb{T}^2 \end{array}$$

Question Is it possible to give a geometric representation of the associated substitutive dynamical system X_σ as a Kronecker map = translation on an abelian compact group?

Yes! (X_σ, S) is measure-theoretically isomorphic to a translation on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

Question How to produce explicitly a fundamental domain?

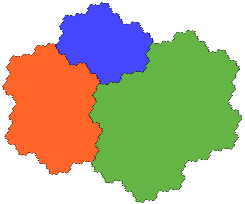
Rauzy fractal G. Rauzy introduced in the 80's a compact set with fractal boundary that tiles the plane which provides a geometric representation of (X_σ, S) $\rightsquigarrow$ Thurston for beta-numeration

Tribonacci dynamics and Tribonacci Kronecker map

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



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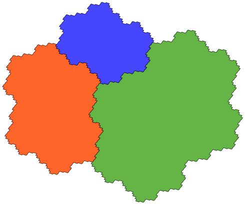

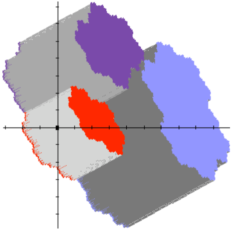

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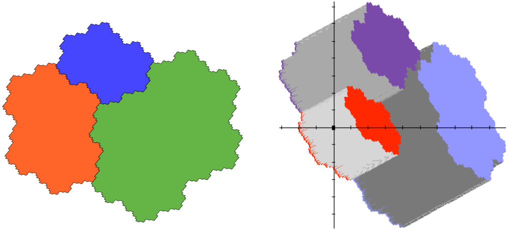


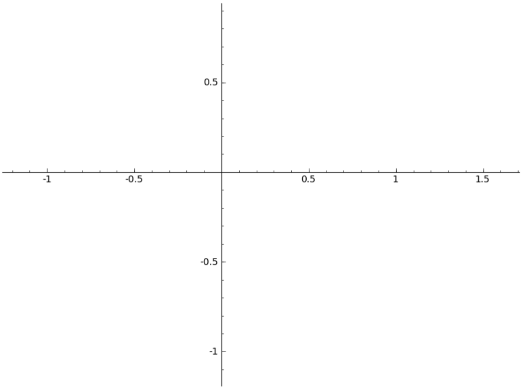
Markov partition for the toral automorphism $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The Rauzy fractal as a geometric representation

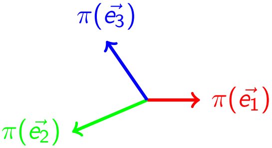
Consider the Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 1$$

$121312112131212131211213\cdots$



π projection along the expanding eigenline onto the contracting plane of the incidence matrix of M_σ



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Consider the Tribonacci substitution

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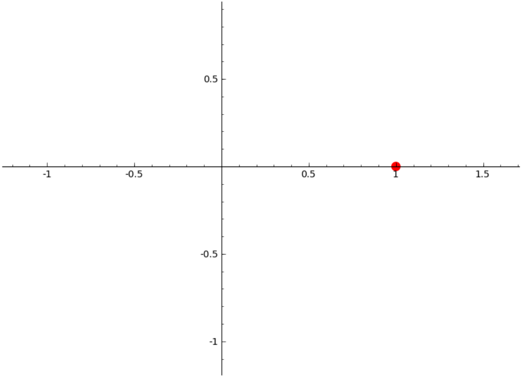
$\pi(\vec{e_1})$



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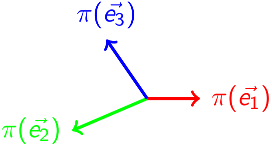

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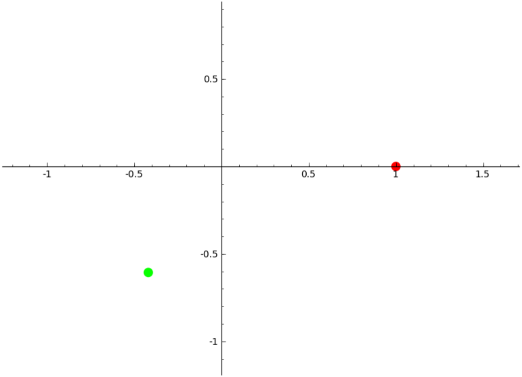

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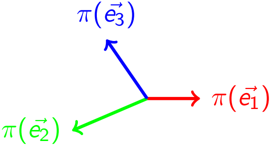

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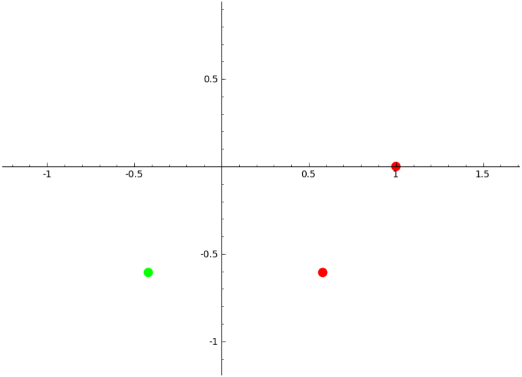

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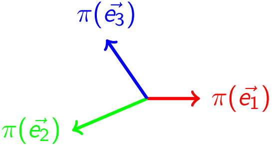

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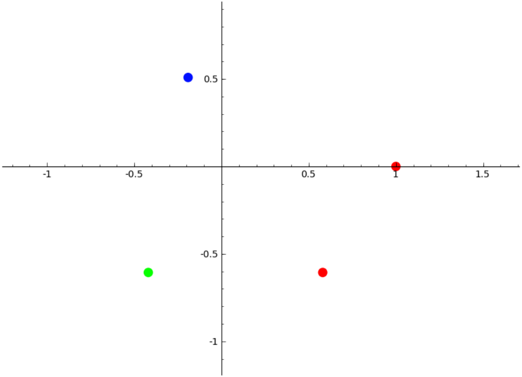

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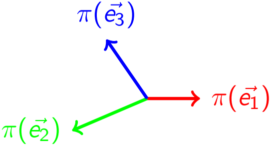

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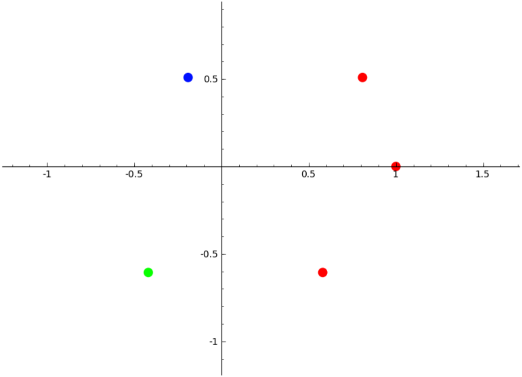

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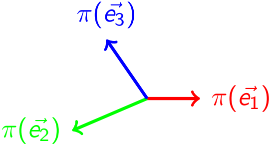

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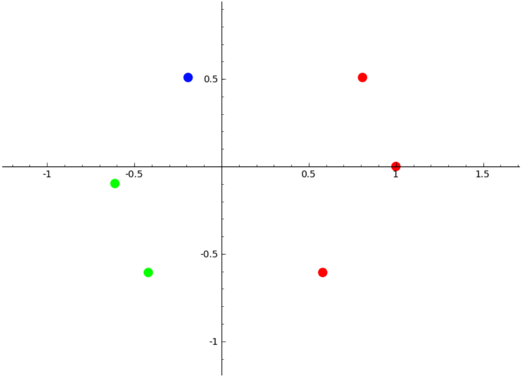

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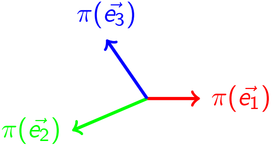

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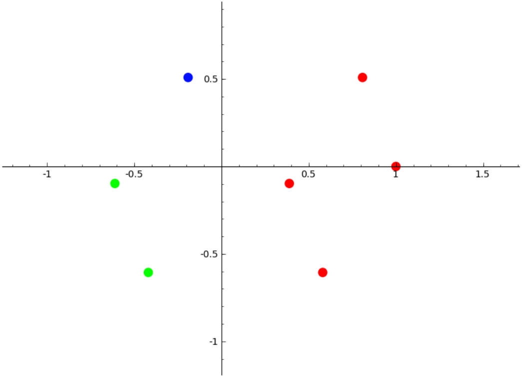

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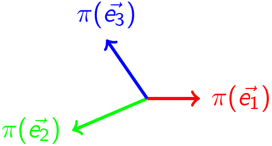

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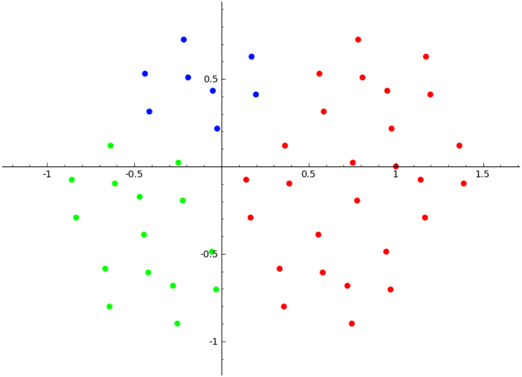

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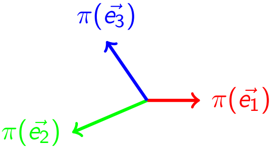

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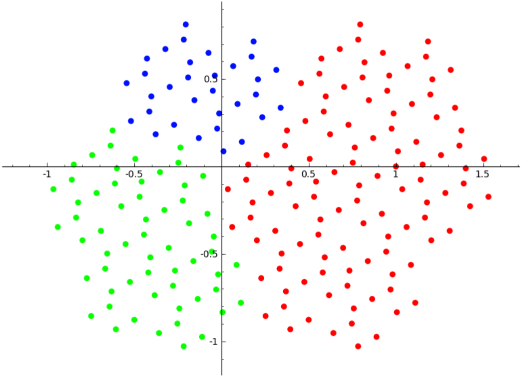

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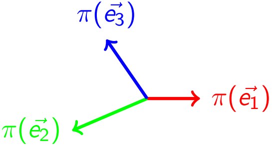

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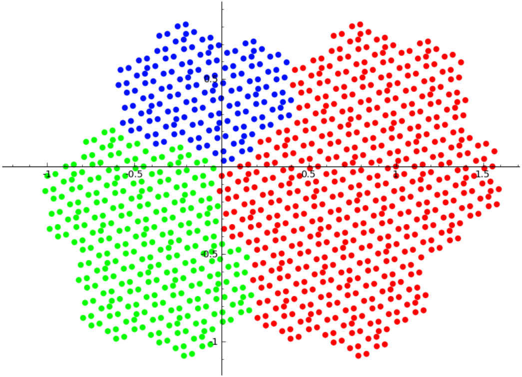

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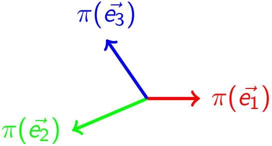

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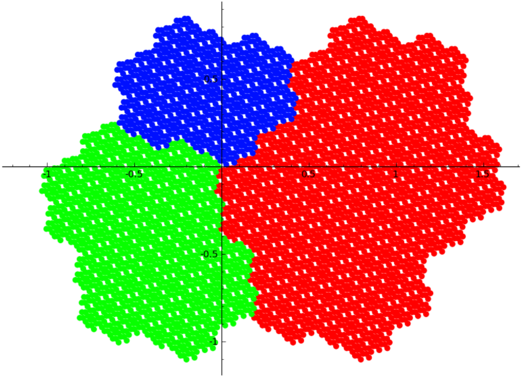

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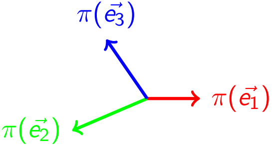

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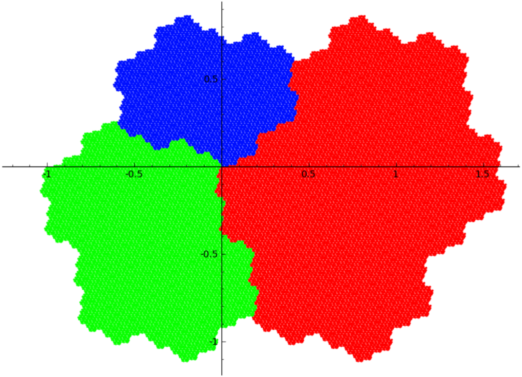

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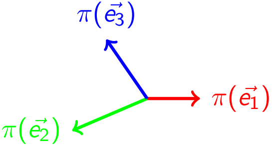
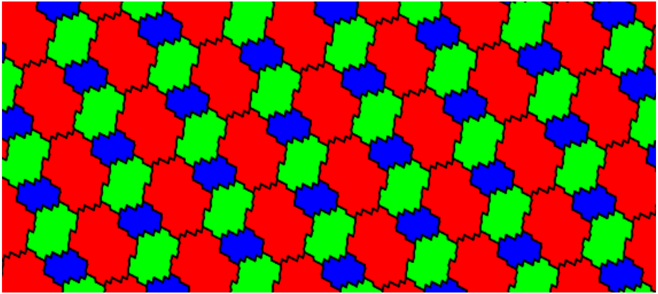


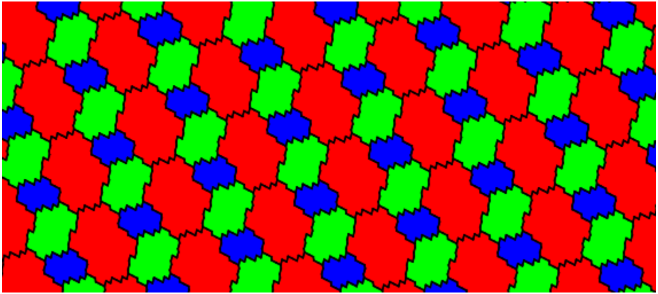
Dynamics of Pisot substitutions

Periodic tiling



Dynamics of Pisot substitutions

Periodic tiling $\longleftrightarrow$ partition of the torus $\mathbb{T}^2$



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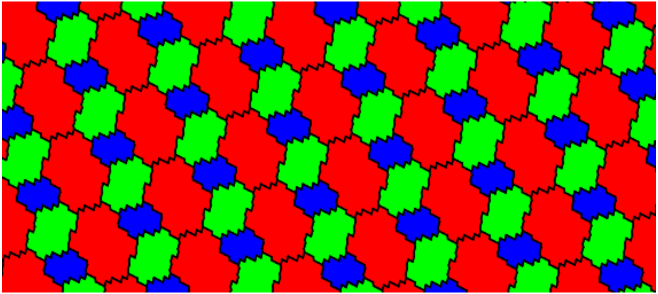

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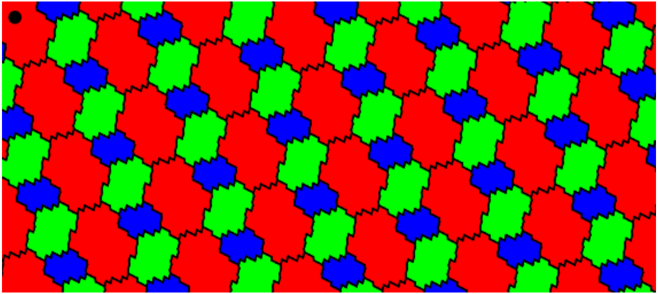

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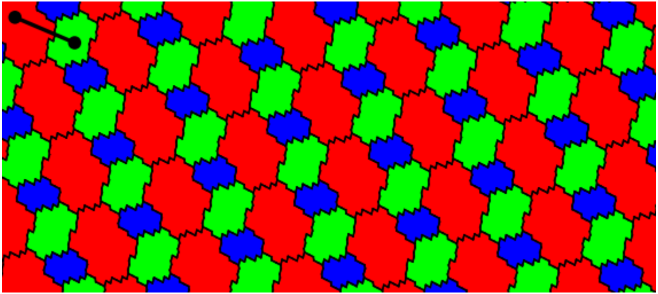

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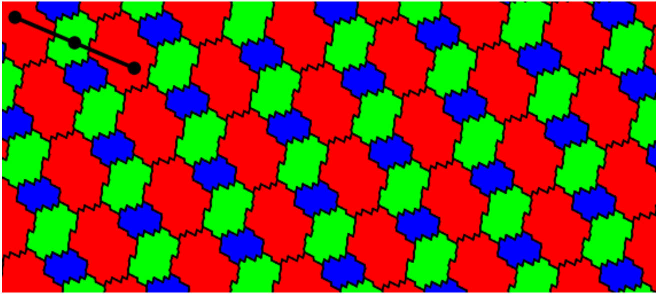

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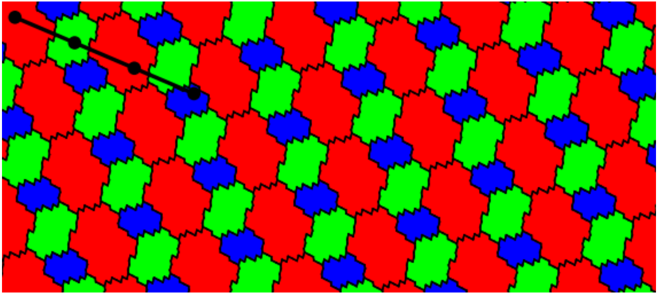

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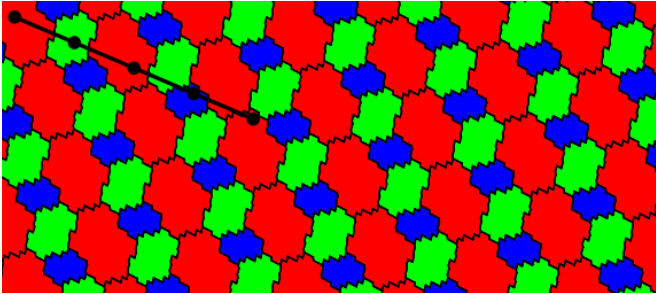

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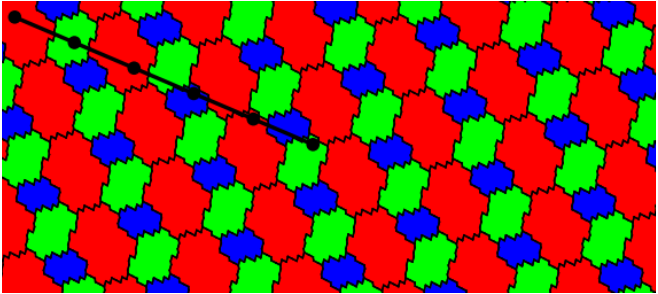

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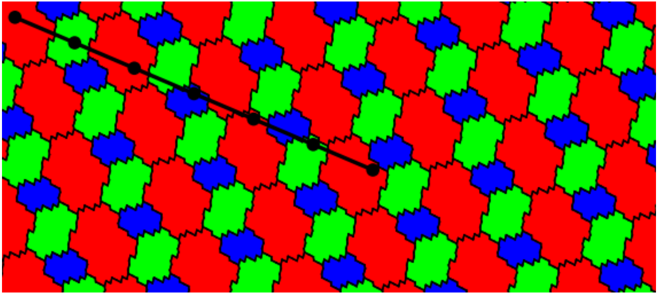

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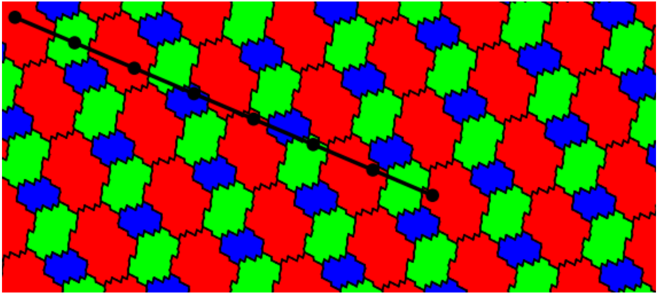

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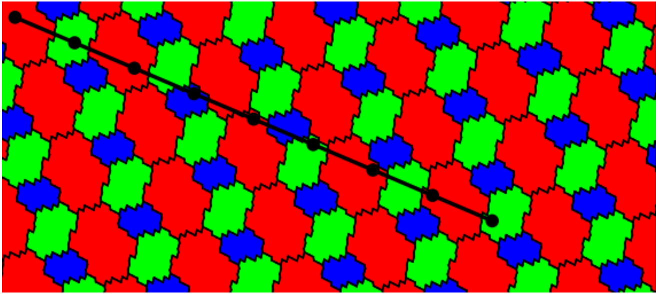

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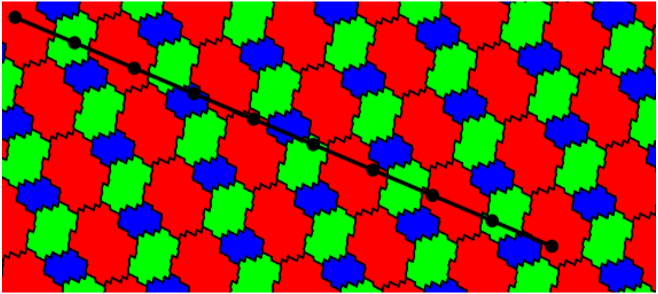

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Why do we get fractals for $d \geq 3$?

- The pieces of the Rauzy fractal are bounded remainder sets
- They produce atoms of Markov partitions for toral automorphisms
- They capture simultaneous approximation properties

Bounded remainder sets and Kronecker sequences

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$

with $1, \alpha_1, \cdots, \alpha_d$ $\mathbb{Q}$-linearly independent

We consider the Kronecker sequence

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

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Bounded remainder set A set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \le n \le N; R_\alpha^n(0) \in X\} - N\mu(X)| \le C$$

Bounded remainder sets

Case $d = 1$

Theorem [Kesten'66] Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

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General dimension d

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Boxes are not bounded remainder sets

It is possible to find polytopes that are bounded remainder sets for any irrational rotation in any dimension
[Haynes-Koivusalo,Grepstad-Lev]

- Renormalization?
- How well can one approximate a box by bounded remainder sets?

Pisot dynamcis

Bounded remainder set A set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \le n \le N; R_\alpha^n(0) \in X\} - N\mu(X)| \le C$$

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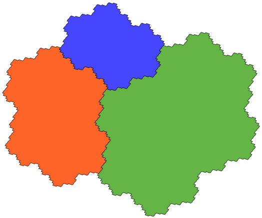
Fact The pieces of the Rauzy fractal are bounded remainder sets



Variations around Rauzy fractals

One can define Rauzy fractals for substitutions over

- Delone sets/cut-and-project schemes [Lee,Moody,Solomyak,Sing,Frettlöh,Baake etc.]
- trees [Bressaud-Jullian]
- on the free group [Arnoux-B.-Hillion-Siegel, Coulbois-Hillion]

and for numeration dynamical systems defined in terms of Pisot numbers

- beta-numeration [Thurston, Akiyama, Ei-Ito-Rao,B.-Siegel, Minervino-Steiner, Barge, etc.]
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and even

- Selmer numbers [Kenyon-Vershik]
- in codimension 2 [Arnoux-Furukado-Harris-Ito]
- Pisot families [Akiyama-Lee, Barge-Stimac-Williams]
- nonalgebraic parameters $\rightsquigarrow$ S-adic Rauzy fractals

S-adic expansions and non-stationary dynamics

Definition An infinite word ω is said S-adic if there exist

- a set of substitutions $\mathcal{S}$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$\omega = \lim_{n\to+\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit
powers of the same substitution= partial quotients

Beyond the Pisot conjecture: S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0,1]^2$, the translation by (α, β) on the torus $\mathbb{T}^2$ admits a symbolic model: the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β)

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Conjecture Every unimodular S-adic Pisot system is measure-theoretically conjugate to a Kronecker translation

Pisot adic dynamics

- Substitutions produce hierarchical ordered structures (infinite words, point sets, tilings) that display strong self-similarity properties
- Substitutions are closely related to induction (first return maps, Rokhlin towers, renormalization etc.)
- Pisot substitutions create a hierarchical structure with a significant amount of long range order
- The Pisot property is a dynamical property
- We can go beyond algebraicity via the S-adic formalism